Pergamon

Int. J. Solids Structures Vol. 33, No. 17, pp. 2417–2436, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00166–2

ELASTIC POSTBUCKLING WITH NONLINEAR CONSTRAINTS

ESBEN BYSKOV, CLAUS DENCKER CHRISTENSEN and KRISTIAN JØRGENSEN

Department of Structural Engineering, Technical University of Denmark, Building 118, DK-2800 Lyngby, Denmark

(*Received* 10 *May* 1995)

Abstract—Koiter's asymptotic theory of initial postbuckling and imperfection sensitivity of elastic structures is expanded such that almost any auxiliary condition can be handled directly. The theory, which employs *Lagrange multiplier techniques*, is valid for strain measures that are quadratic in the displacements, i.e. it is exact for *Lagrangian* strain measures, and auxiliary conditions and loadings that are quartic in their arguments. Although the general formulas are rather lengthy and complicated in the general case they simplify considerably in almost all cases of practical interest. Frequently the formulas are only slightly more complex than in the equivalent situation without auxiliary conditions.

The method is applied to the example of an elastic inextensible circular ring but is suitable for all elastic structures which comply with the above mentioned requirements regarding strain measure and auxiliary conditions. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

In order to obtain an accurate structural analysis it is sometimes necessary to impose constraints such as inextensibility. In other situations, it may be essential to handle numerical difficulties such as membrane locking. For these purposes several different approaches have been applied with success, e.g. *reduced integration* (Noor and Peters, 1981), (Belytschko *et al.*, 1985), *mixed methods* (Noor and Peters, 1981), (Belytschko *et al.*, 1985) and (Stolarski and Belytschko, 1983), *mode decomposition methods* (Belytschko *et al.*, 1985), (Stolarski and Belytschko, 1983) and (Mau and El-Mabsout, 1989), and *Lagrange multiplier techniques* (Byskov, 1989b).

Here, we concentrate on asymptotic analysis of postbuckling and imperfection sensitivity in the spirit of Koiter (1945), Budiansky and Hutchinson (1964), Fitch (1968), and Budiansky (1974) and establish a general method that is capable of handling auxiliary conditions and load terms which are quartic in their arguments and strain measures which are quadratic in their arguments. For the purpose of extending Koiter's theory only the Lagrange multiplier technique seems to constitute a viable alternative. The reason is that the other approaches mentioned above are particularly well suited for certain special problems and therefore of a less general nature.

Inextensibility may pose severe problems in actual applications of Koiter's theory, see e.g. Budiansky (1974) or Sills and Budiansky (1978), where the problem of buckling and postbuckling of an elastic ring is solved. Except for the study by Sills and Budiansky (1978) it seems that inextensibility has been handled in various *ad hoc* fashions, although Budiansky (1974) very briefly outlines a procedure like ours. Another important issue associated with numerical postbuckling studies and, more broadly geometrically nonlinear studies, is the handling of locking, be it membrane, shear, or bending locking in curved or straight finite elements. For these purposes, our method is very efficient, as will be shown in a later article.

Examples of auxiliary conditions

For convenience and clarity we give some examples of auxiliary conditions below.

Inextensibility. For the complete circular ring treated by Sills and Budiansky (1978) inextensibility is given as:

$$0 = w + \frac{\mathrm{d}v}{\mathrm{d}\alpha} + \frac{1}{2}\left(\left(w + \frac{\mathrm{d}v}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}w}{\mathrm{d}\alpha} - v\right)^2\right) \tag{1}$$

where w and v are the nondimensional outward and axial displacement component, respectively, and α is the sectorial angle. In most analyses a condition like (1) is difficult to fulfill explicitly, and indirect means are more feasible.

Locking in general. In conventional nonlinear finite element analyses there is a tendency to develop nonlinear membrane or bending locking that causes inaccurate values of the membrane or bending strains and stresses. The reason is that the two terms in the expression in the strain-displacement relation, see (12a), usually are approximated by polynomials of different degree:

$$\varepsilon = \mathscr{L}_1(u) + \tfrac{1}{2}\mathscr{L}_2(u) \tag{2}$$

Here, ε denotes the generalized strains, u designates the generalized displacements, and $\mathscr{L}_1$ is a linear, and $\mathscr{L}_2$ a quadratic operator, respectively. While the first term in the right hand side of (2) is likely to vary fairly smoothly, the second is usually characterized by rapid variations. This means that in most finite element analyses the nonlinear strains are poorly described by the sum of the two terms. It may, however, be worthwhile noticing that the two terms are computed at the same time and that the strain energy is given by ε, not the individual terms. Therefore, it seems possible that ε may behave less erratically than the two terms individually.

Locking in postbuckling studies. For a symmetric structure the postbuckling strain ε_2 is given as, see (21b):

$$\varepsilon_2 = \mathscr{L}_1(u_2) + \mathscr{L}_{11}(u_c, u_2) + \tfrac{1}{2}\mathscr{L}_2(u_1) \tag{3}$$

where u_c is the prebuckling displacement field computed at the classical critical load, characterized by the value λ_c of the load parameter λ, u_1 is the buckling displacement field, u_2 is the postbuckling displacement field, and $\mathscr{L}_{11}$ is a bilinear operator which is derived from $\mathscr{L}_2$, see (12a) and (83a).

Since u_1 is furnished by the buckling problem and therefore is given *before* the postbuckling problem is established, the terms $\mathscr{L}_1(u_2)$ and $\mathscr{L}_{11}(u_c, u_2)$ must be able to accommodate all possible rapid variations in the term $\frac{1}{2}\mathscr{L}_2(u_1)$. Apparently because of this lack of freedom, nonlinear membrane locking seems to be even more severe in postbuckling studies than in other geometrically nonlinear problems.

In earlier studies, see e.g. Byskov (1989b) or Byskov (1989a), the method of Lagrange multipliers has been applied to postbuckling problems with *linear* prebuckling, where membrane locking only occurs in the computation of the postbuckling strains and stresses. Analyses of structures that exhibit *nonlinear* prebuckling, e.g. arches with point loads, necessitates taking membrane locking into account in all steps of the computation. For such purposes our extension to Koiter's theory proves to be convenient and efficient because it entails application of the Lagrange multiplier technique from the outset rather than treating the prebuckling, the buckling, and the postbuckling problems individually.

2. THE PRINCIPLE OF VIRTUAL DISPLACEMENTS WITH CONSTRAINTS

A *Modified Principle of Virtual Displacements* with nonlinear loading terms and *Lagrange Multiplier* terms may be written:

$$\sigma \cdot \delta\varepsilon(u) = \lambda\delta B(u) + \delta(\eta \cdot C(u)) \tag{4}$$

Here, u denotes the displacement field; ε is the strain field, σ its corresponding stress field given as $H(\varepsilon)$, where H is a linear operator; δ designates variation; λ is a scalar load parameter; B is a (nonlinear) loading functional; C contains the appropriate constraints; η is the Lagrange multiplier field corresponding to C; and a dot $(\cdot)$, according to the Budiansky-Hutchinson notation (Budiansky and Hutchinson, 1964), indicates an inner product. The assumption of linear (hyper)elasticity implies the reciprocity relation:

$$H(\varepsilon(u_a))\varepsilon(u_b) = H(\varepsilon(u_b))\varepsilon(u_a) \tag{5}$$

Ultimately, σ and ε in (4) may be expressed in terms of the displacements, which then constitute the primary unknowns. Occasionally, the Lagrange multiplier field may be eliminated before the final set of global equations is formulated, which can reduce the computational expense. Except for notational differences, (4) is equivalent to an unnumbered formula in (Budiansky, 1974), p. 30. While Budiansky exploits his formula in two examples, he does not derive a set of general equations like the ones below.

3. FUNDAMENTAL PATH, BUCKLING AND POSTBUCKLING

Let subscript $_0$ indicate prebuckling quantities, then the fundamental path, given by u_0, σ_0, ε_0 and η_0, can be found from the modified principle of virtual displacements (4).

We establish the modified principle of virtual displacements (4) on the fundamental path and the bifurcated path, respectively. Let the load level λ be the same in both variational equations and let it approach the classical critical value λ_c of λ. By subtracting the two principles of virtual work we may then derive variational statements that govern the buckling mode u_1, λ_c, and the initial postbuckling behavior, see below.

3.1. *Perturbation expansion*

For λ close to λ_c, λ and u may be expanded in perturbation series, see e.g. (Budiansky, 1974) or (Hutchinson, 1974):

$$\lambda/\lambda_c = 1 + a\xi + b\xi^2 + c\xi^3 + O(\xi^4) \tag{6}$$

and

$$u = u_0(\lambda) + \xi u_1 + \xi^2 u_2 + \xi^3 u_3 + O(\xi^4) \tag{7}$$

where ξ is the perturbation parameter, which we later shall identify as the buckling mode amplitude.

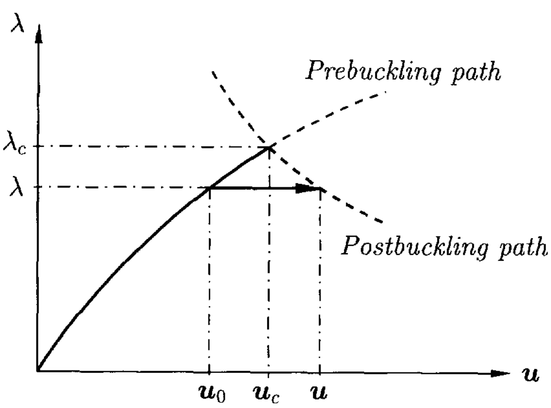

Fig. 1. Fundamental and bifurcated paths.

The fields σ, ε and η, which we for brevity and convenience collectively denote p, may be expanded as:

$$p = p_0(\lambda) + \xi p_1 + \xi^2 p_2 + \xi^3 p_3 + O(\xi^4) \tag{8}$$

with the variation δp expanded in a similar fashion:

$$\delta p = \delta p_0(\lambda) + \xi \delta p_1 + \xi^2 \delta p_2 + \xi^3 \delta p_3 + O(\xi^4) \tag{9}$$

Note that δp_i is defined as the field which is the coefficient to ξ^i in the expansion of δp and that δp_i may not be computed as a variation of p_i.

The fields p are either quartic in u or expanded to fourth order in u in order to insure that the theory is able to handle problems with a high degree of nonlinearity:

$$p(u) = \mathscr{P}_1(u) + \tfrac{1}{2}\mathscr{P}_2(u) + \tfrac{1}{3}\mathscr{P}_3(u) + \tfrac{1}{4}\mathscr{P}_4(u) \tag{10}$$

Rules for the operators $\mathscr{P}_i$ are given in Appendix A and provide:

$$\delta p(u) = \mathscr{P}_1(\delta u) + \mathscr{P}_{11}(\delta u, u) + \mathscr{P}_{12}(\delta u, u) + \mathscr{P}_{13}(\delta u, u) \tag{11}$$

To be specific, for ε, B, and C the expansion (10) is interpreted as:

$$\begin{aligned}
\varepsilon(u) &= \mathscr{L}_1(u) + \tfrac{1}{2}\mathscr{L}_2(u) \\
B(u) &= \mathscr{B}_1(u) + \tfrac{1}{2}\mathscr{B}_2(u) + \tfrac{1}{3}\mathscr{B}_3(u) + \tfrac{1}{4}\mathscr{B}_4(u) \\
C(u) &= \mathscr{C}_1(u) + \tfrac{1}{2}\mathscr{C}_2(u) + \tfrac{1}{3}\mathscr{C}_3(u) + \tfrac{1}{4}\mathscr{C}_4(u)
\end{aligned} \tag{12}$$

where it is noted that Lagrange strains are quadratic in the displacements and are therefore represented exactly by our theory.

3.2. *Taylor expansion*

In order to investigate the prebuckling path close to the bifurcation point we expand all fields in λ:

$$p_0(\lambda) = p_c + (\lambda - \lambda_c)p_c' + \tfrac{1}{2}(\lambda - \lambda_c)^2 p_c'' + \tfrac{1}{6}(\lambda - \lambda_c)^3 p_c''' + O((\lambda - \lambda_c)^4) \tag{13}$$

Here, $p_0(\lambda)$ denotes any field on the prebuckling path, and a prime indicates differentiation with respect to λ:

$$(\ )' \equiv \frac{\partial(\ )}{\partial \lambda} \tag{14}$$

and subscript $_c$ indicates that the value of the quantity is taken at $\lambda = \lambda_c$. It may be worthwhile noticing that, although ξ is a quantity that is defined on the *postbuckling* path, it is also used as a perturbation parameter on the *prebuckling* path. This, however, does not entail any inconsistencies in that the introduction of ξ on the prebuckling path simply may be viewed as a change of variables.

Insert λ given as (6) into (8), utilize the expansion (13), and gather terms with ξ of the same order to obtain the following expressions:

$$p = p_c + \xi p_1^* + \xi^2 p_2^* + \xi^3 p_3^* + O(\xi^4) \tag{15}$$

and similarly

$$\delta p = \delta p_c + \xi \delta p_1^* + \xi^2 \delta p_2^* + \xi^3 \delta p_3^* + O(\xi^4) \tag{16}$$

and

$$u = u_c + \xi u_1^* + \xi^2 u_2^* + \xi^3 u_3^* + O(\xi^4) \tag{17}$$

Here, upper index * is introduced to differentiate from the previous expansions, e.g. (7) and (8), and the fields p_i^*, δp_i^* and u_i^*, $i \in [1, 3]$, are independent of λ and ξ. The expressions for p_i^*, δp_i^* and u_i^*, which follow from (6)–(9), (13) and (15)–(17), are:

$$\begin{aligned}
p_1^* &= a\lambda_c p_c' + p_1 \\
p_2^* &= b\lambda_c p_c' + \tfrac{1}{2}a^2\lambda_c^2 p_c'' + p_2 \\
p_3^* &= c\lambda_c p_c' + ab\lambda_c^2 p_c'' + \tfrac{1}{6}a^3\lambda_c^3 p_c''' + p_3
\end{aligned} \tag{18}$$

$$\begin{aligned}
\delta p_1^* &= a\lambda_c \delta p_c' + \delta p_1 \\
\delta p_2^* &= b\lambda_c \delta p_c' + \tfrac{1}{2}a^2\lambda_c^2 \delta p_c'' + \delta p_2 \\
\delta p_3^* &= c\lambda_c \delta p_c' + ab\lambda_c^2 \delta p_c'' + \tfrac{1}{6}a^3\lambda_c^3 \delta p_c''' + \delta p_3
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
u_1^* &= a\lambda_c u_c' + u_1 \\
u_2^* &= b\lambda_c u_c' + \tfrac{1}{2}a^2\lambda_c^2 u_c'' + u_2 \\
u_3^* &= c\lambda_c u_c' + ab\lambda_c^2 u_c'' + \tfrac{1}{6}a^3\lambda_c^3 u_c''' + u_3
\end{aligned} \tag{20}$$

In Appendix B it is shown that p_1, p_2 and p_3 may be written:

$$\begin{aligned}
p_1 = {} & \mathscr{P}_1(u_1) + \mathscr{P}_{11}(u_1, u_c) + \mathscr{P}_{12}(u_1, u_c) + \mathscr{P}_{13}(u_1, u_c) \\
p_2 = {} & + a\lambda_c[\mathscr{P}_{11}(u_c', u_1) + 2\mathscr{P}_{111}(u_c, u_c', u_1) + 3\mathscr{P}_{112}(u_c', u_1, u_c)] \\
& + \mathscr{P}_1(u_2) + \tfrac{1}{2}\mathscr{P}_2(u_1) + \mathscr{P}_{11}(u_c, u_2) + \mathscr{P}_{12}(u_c, u_1) \\
& + \mathscr{P}_{12}(u_2, u_c) + \tfrac{3}{2}\mathscr{P}_{22}(u_c, u_1) + \mathscr{P}_{13}(u_2, u_c) \\
p_3 = {} & + a\lambda_c[+\mathscr{P}_{11}(u_c', u_2) + 2\mathscr{P}_{111}(u_c, u_c', u_2) + \mathscr{P}_{12}(u_c', u_1) \\
& + 3\mathscr{P}_{112}(u_c', u_2, u_c) + 3\mathscr{P}_{112}(u_c, u_c', u_1)] \\
& + b\lambda_c[\mathscr{P}_{11}(u_c', u_1) + 2\mathscr{P}_{111}(u_c, u_c', u_1) + 3\mathscr{P}_{112}(u_c', u_1, u_c)] \\
& + \tfrac{1}{2}(a\lambda_c)^2[+\mathscr{P}_{11}(u_c'', u_1) + 2\mathscr{P}_{111}(u_c, u_c'', u_1) + 2\mathscr{P}_{12}(u_1, u_c') \\
& + 3\mathscr{P}_{112}(u_c'', u_1, u_c) + 6\mathscr{P}_{112}(u_c, u_1, u_c')] \\
& + \mathscr{P}_1(u_3) + \mathscr{P}_{11}(u_1, u_2) + \mathscr{P}_{11}(u_c, u_3) + 2\mathscr{P}_{111}(u_c, u_1, u_2) \\
& + \mathscr{P}_{12}(u_3, u_c) + \tfrac{1}{3}\mathscr{P}_3(u_1) + 3\mathscr{P}_{112}(u_1, u_2, u_c) \\
& + \mathscr{P}_{13}(u_3, u_c) + \mathscr{P}_{13}(u_c, u_1)
\end{aligned} \tag{21}$$

In Appendix C it is shown that δp_1, δp_2 and δp_3 may be written:

$$
\begin{aligned}
\delta p_1 &= \mathscr{P}_{11}(\delta u, u_1) + 2\mathscr{P}_{111}(\delta u, u_c, u_1) + 3\mathscr{P}_{112}(\delta u, u_1, u_c) \\
\delta p_2 &= + a\lambda_c[2\mathscr{P}_{111}(\delta u, u'_c, u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u'_c, u_1)] \\
&\quad + \mathscr{P}_{11}(\delta u, u_2) + \mathscr{P}_{12}(\delta u, u_1) + 2\mathscr{P}_{111}(\delta u, u_c, u_2) + 3\mathscr{P}_{112}(\delta u, u_c, u_1) \\
&\quad + 3\mathscr{P}_{112}(\delta u, u_2, u_c) \\
\delta p_3 &= + a\lambda_c[2\mathscr{P}_{111}(\delta u, u'_c, u_2) + 6\mathscr{P}_{1111}(\delta u, u_c, u'_c, u_2) + 3\mathscr{P}_{112}(\delta u, u'_c, u_1)] \\
&\quad + b\lambda_c[2\mathscr{P}_{111}(\delta u, u'_c, u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u'_c, u_1)] \\
&\quad + \tfrac{1}{2}a^2\lambda_c^2[2\mathscr{P}_{111}(\delta u, u''_c, u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u''_c, u_1) + 6\mathscr{P}_{112}(\delta u, u_1, u'_c)] \\
&\quad + \mathscr{P}_{11}(\delta u, u_3) + 2\mathscr{P}_{111}(\delta u, u_1, u_2) + 2\mathscr{P}_{111}(\delta u, u_c, u_3) \\
&\quad + 6\mathscr{P}_{1111}(\delta u, u_c, u_1, u_2) + 3\mathscr{P}_{112}(\delta u, u_3, u_c) + \mathscr{P}_{13}(\delta u, u_1)
\end{aligned} \tag{22}
$$

3.3 *Principle of virtual displacements in asymptotic form*

The terms in the principle of virtual displacements (4) are introduced by their asymptotic expansions (15)–(20):

$$
\begin{aligned}
&(\sigma_c + \xi\sigma_1^* + \xi^2\sigma_2^* + \xi^3\sigma_3^* + O(\xi^4)) \cdot (\delta\varepsilon_c + \xi\delta\varepsilon_1^* + \xi^2\delta\varepsilon_2^* + \xi^3\delta\varepsilon_3^* + O(\xi^4)) \\
&\quad = (1 + \xi a + \xi^2 b + \xi^3 c + O(\xi^4))\lambda_c(\delta B_c + \xi\delta B_1^* + \xi^2\delta B_2^* + \xi^3\delta B_3^* + O(\xi^4)) \\
&\qquad + (C_c + \xi C_1^* + \xi^2 C_2^* + \xi^3 C_3^* + O(\xi^4)) \cdot \delta\eta \\
&\qquad + (\eta_c + \xi\eta_1^* + \xi^2\eta_2^* + \xi^3\eta_3^*) \cdot (\delta C_c + \xi\delta C_1^* + \xi^2\delta C_2^* + \xi^3\delta C_3^* + O(\xi^4))
\end{aligned} \tag{23}
$$

Since the asymptotic expansion is assumed to be valid for any (small) value of ξ we may collect terms of like order in ξ and establish variational equations for the eigenvalue problem and the boundary value problems of increasing order in ξ. In this way we get the variational eqns (24)–(27) below.

Zeroth order problem at bifurcation.

$$
0 = \sigma_c \cdot \delta\varepsilon_c - \lambda_c\delta B_c - C_c \cdot \delta\eta - \eta_c \cdot \delta C_c \tag{24}
$$

which, of course, is nothing else than the principle of virtual displacements (4) on the prebuckling path at bifurcation.

First order problem. The first order problem, i.e. the eigenvalue problem is:

$$
0 = \sigma_1^* \cdot \delta\varepsilon_c + \sigma_c \cdot \delta\varepsilon_1^* - \lambda_c\delta B_1^* - a\lambda_c\delta B_c - C_1^* \cdot \delta\eta - \eta_c \cdot \delta C_1^* - \eta_1^* \cdot \delta C_c \tag{25}
$$

Second order problem. This problem, which sometimes is also referred to as the first postbuckling problem is:

$$
\begin{aligned}
0 = {} & \sigma_c \cdot \delta\varepsilon_2^* + \sigma_1^* \cdot \delta\varepsilon_1^* + \sigma_2^* \cdot \delta\varepsilon_c - \lambda_c\delta B_2^* - a\lambda_c\delta B_1^* \\
& - b\lambda_c\delta B_c - C_2^* \cdot \delta\eta - \eta_c \cdot \delta C_2^* - \eta_1^* \cdot \delta C_1^* - \eta_2^* \cdot \delta C_c
\end{aligned} \tag{26}
$$

Third order problem. This is occasionally denoted the second postbuckling problem:

$$
\begin{aligned}
0 = {} & \sigma_c \cdot \delta\varepsilon_3^* + \sigma_1^* \cdot \delta\varepsilon_2^* + \sigma_2^* \cdot \delta\varepsilon_1^* + \sigma_3^* \cdot \delta\varepsilon_c - \lambda_c\delta B_3^* \\
& - a\lambda_c\delta B_2^* - b\lambda_c\delta B_1^* - c\lambda_c\delta B_c - C_3^* \cdot \delta\eta - \eta_c \cdot \delta C_3^* - \eta_1^* \cdot \delta C_2^* - \eta_2^* \cdot \delta C_1^* - \eta_3^* \cdot \delta C_c
\end{aligned} \tag{27}
$$

The postbuckling constants a, b and c are later eliminated from (25), (26) and (27), respectively.

In order to determine a we need (26), and to compute b the expression (27) is a prerequisite, but we shall not carry the expansions to third order except to the extent which is necessary to compute b by use of (27), and thus (27) is of temporary value only.

4. PREBUCKLING

The boundary value problem for the fundamental path is obtained by inserting (10) and (11) in (4) and noting that $u = u_0$ in prebuckling:

$$\begin{aligned} H[\mathscr{L}_1(u_0)+\tfrac{1}{2}\mathscr{L}_2(u_0)]\cdot[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_0)] \\ = \lambda[\mathscr{B}_1(\delta u)+\mathscr{B}_{11}(\delta u,u_0)+\mathscr{B}_{12}(\delta u,u_0)+\mathscr{B}_{13}(\delta u,u_0)] \\ +\delta\eta\cdot[\mathscr{C}_1(u_0)+\tfrac{1}{2}\mathscr{C}_2(u_0)+\tfrac{1}{3}\mathscr{C}_3(u_0)+\tfrac{1}{4}\mathscr{C}_4(u_0)] \\ +\eta_0\cdot[\mathscr{C}_1(\delta u)+\mathscr{C}_{11}(\delta u,u_0)+\mathscr{C}_{12}(\delta u,u_0)+\mathscr{C}_{13}(\delta u,u_0)] \end{aligned} \tag{28}$$

5. BUCKLING

Insert (18) and (19) in the buckling problem (25) and utilize the first derivative of the modified principle of virtual displacements at bifurcation (102) to eliminate all terms containing the unknown first order postbuckling constant a and get an eigenvalue problem to determine λ_c and u_1:

$$0 = \sigma_c\cdot\delta\varepsilon_1+\sigma_1\cdot\delta\varepsilon_c-\eta_c\cdot\delta C_1-\eta_1\cdot\delta C_c-C_1\cdot\delta\eta-\lambda_c\delta B_1 \tag{29}$$

When the operator expansions (21) and (22) are exploited (29) yields:

$$0 = \mathscr{E}_{11}(u_1,\delta u) \tag{30}$$

with the functional $\mathscr{E}_{11}(u_1, \delta u)$ defined by:

$$\begin{aligned} \mathscr{E}_{11}(u_1,\delta u) \equiv\ & -\sigma_c\cdot\mathscr{L}_{11}(\delta u,u_1) \\ & -H[\mathscr{L}_1(u_1)+\mathscr{L}_{11}(u_c,u_1)]\cdot[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_c)] \\ & +\eta_c\cdot[\mathscr{C}_{11}(\delta u,u_1)+2\mathscr{C}_{111}(\delta u,u_c,u_1)+3\mathscr{C}_{112}(\delta u,u_1,u_c)] \\ & +\eta_1\cdot[\mathscr{C}_1(\delta u)+\mathscr{C}_{11}(\delta u,u_c)+\mathscr{C}_{12}(\delta u,u_c)+\mathscr{C}_{13}(\delta u,u_c)] \\ & +\delta\eta\cdot[\mathscr{C}_1(u_1)+\mathscr{C}_{11}(u_1,u_c)+\mathscr{C}_{12}(u_1,u_c)+\mathscr{C}_{13}(u_1,u_c)] \\ & +\lambda_c[\mathscr{B}_{11}(\delta u,u_1)+2\mathscr{B}_{111}(\delta u,u_c,u_1)+3\mathscr{B}_{112}(\delta u,u_1,u_c)] \end{aligned} \tag{31}$$

By use of the reciprocity relation (5) it is observed that $\mathscr{E}_{11}$ is symmetric:

$$\mathscr{E}_{11}(u,v) = \mathscr{E}_{11}(v,u)\ \forall\ \text{kinematically admissible}\ (u,v) \tag{32}$$

Note that a does not enter (30) and that u_1 only enters linearly, as expected.

6. POSTBUCKLING

In order to carry out a second order analysis, i.e. an analysis up to ξ^2, see (6), we derive variational equations for the first and second postbuckling problems which determine u_2 and u_3, respectively. Although we do not intend to compute u_3, as mentioned above, it proves necessary to establish the second postbuckling problem, which might be used to find u_3, in order to determine the postbuckling constant b.

6.1. *First postbuckling problem*

In order to determine the first order postbuckling constant a we exploit the first postbuckling problem (26) and utilize the the first and second derivatives of the modified principle of virtual displacements, (102) and (103), respectively, to eliminate the second order postbuckling constant b:

$$0 = a\lambda_c[-\delta B_1 - \eta_c' \cdot \delta C_1 - \eta_1 \cdot \delta C_c' + \sigma_c' \cdot \delta\varepsilon_1 + \sigma_1 \cdot \delta\varepsilon_c'] \\ -[\lambda_c \delta B_2 + C_2 \cdot \delta\eta + \eta_c \cdot \delta C_2 + \eta_2 \cdot \delta C_c + \eta_1 \cdot \delta C_1 - \sigma_c \cdot \delta\varepsilon_2 - \sigma_2 \cdot \delta\varepsilon_c - \sigma_1 \cdot \delta\varepsilon_1] \quad (33)$$

which contains the unknown postbuckling field u_2 as well as a. After introduction of p_i and δp_i given by (21) and (22), respectively, we collect terms that contain u_2 on the left hand side and get:

$$\mathscr{E}_{11}(u_2, \delta u) = \mathscr{F}_1^1(\delta u) + a\lambda_c \mathscr{F}_1^2(\delta u) \quad (34)$$

which may be solved for u_2 when a is known, see (39) below, $\mathscr{E}_{11}$ is given by (31), and the right hand side functionals are defined by:

$$\begin{aligned} \mathscr{F}_1^1(\delta u) = & -\lambda_c[\mathscr{B}_{12}(\delta u, u_1) + 3\mathscr{B}_{112}(\delta u, u_c, u_1)] \\ & -\delta\eta \cdot [\tfrac{1}{2}\mathscr{C}_2(u_1) + \mathscr{C}_{12}(u_c, u_1) + \tfrac{3}{2}\mathscr{C}_{22}(u_c, u_1)] \\ & -\eta_c \cdot [\mathscr{C}_{12}(\delta u, u_1) + 3\mathscr{C}_{112}(\delta u, u_c, u_1)] \\ & -\eta_1 \cdot [\mathscr{C}_{11}(\delta u, u_1) + 2\mathscr{C}_{111}(\delta u, u_c, u_1) + 3\mathscr{C}_{112}(\delta u, u_1, u_c)] \\ & + H[\tfrac{1}{2}\mathscr{L}_2(u_1)] \cdot [\mathscr{L}_1(\delta u) + \mathscr{L}_{11}(\delta u, u_c)] + \sigma_1 \cdot \mathscr{L}_{11}(\delta u, u_1) \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathscr{F}_1^2(\delta u) = & +\sigma_c' \cdot \mathscr{L}_{11}(\delta u, u_1) + \sigma_1 \cdot \mathscr{L}_{11}(\delta u, u_c') + H[\mathscr{L}_{11}(u_c', u_1)] \cdot [\mathscr{L}_1(\delta u) + \mathscr{L}_{11}(\delta u, u_c)] \\ & -[\mathscr{B}_{11}(\delta u, u_1) + 2\mathscr{B}_{111}(\delta u, u_c, u_1) + 3\mathscr{B}_{112}(\delta u, u_1, u_c)] \\ & -\eta_c' \cdot [\mathscr{C}_{11}(\delta u, u_1) + 2\mathscr{C}_{111}(\delta u, u_c, u_1) + 3\mathscr{C}_{112}(\delta u, u_1, u_c)] \\ & -\eta_c \cdot [2\mathscr{C}_{111}(\delta u, u_c', u_1) + 6\mathscr{C}_{1111}(\delta u, u_c, u_c', u_1)] \\ & -\eta_1 \cdot [\mathscr{C}_{11}(\delta u, u_c') + 2\mathscr{C}_{111}(\delta u, u_c, u_c') + 3\mathscr{C}_{112}(\delta u, u_c', u_c)] \\ & -\delta\eta \cdot [\mathscr{C}_{11}(u_1, u_c') + 2\mathscr{C}_{111}(u_1, u_c, u_c') + 3\mathscr{C}_{112}(u_1, u_c', u_c)] \\ & -\lambda_c[2\mathscr{B}_{111}(\delta u, u_c', u_1) + 6\mathscr{B}_{1111}(\delta u, u_c, u_c', u_1)] \end{aligned} \quad (36)$$

To obtain an expression for a, the buckling problem (30) with $\delta u = u_2$ is subtracted from the first postbuckling problem (34) with $\delta u = u_1$. With $\delta u = u_2$ the buckling problem (30) yields:

$$0 = \mathscr{E}_{11}(u_1, u_2) \quad (37)$$

The first postbuckling problem (34) with $\delta u = u_1$ provides:

$$\mathscr{E}_{11}(u_2, u_1) = \mathscr{F}_1^1(u_1) + a\lambda_c \mathscr{F}_1^2(u_1) \quad (38)$$

When we subtract (37) from (38) and utilize the reciprocity relation (32) we may get the expression (39) for the determination of a:

$$a\lambda_c = -\frac{\mathscr{F}_1^1(u_1)}{\mathscr{F}_1^2(u_1)} \tag{39}$$

Since the second order field u_2 does not enter, eqn (39) can be used to find the first order postbuckling constant a.

6.2. *Second postbuckling problem*

Analogous to the procedure utilized in Section 6.1 we use the first, second and third derivatives of the principle of virtual displacements (102), (103) and (104), respectively, to eliminate all terms containing the third order postbuckling constant c from the second postbuckling problem (27) to get a formula for the second order postbuckling constant b:

$$\begin{aligned} 0 = {} & +b\lambda_c[-\delta B_1 - \eta_c' \cdot \delta C_1 - \eta_1 \cdot \delta C_c' + \sigma_c' \cdot \delta\varepsilon_1 + \sigma_1 \cdot \delta\varepsilon_c'] \\ & -\tfrac{1}{2}(a\lambda_c)^2[\eta_c'' \cdot \delta C_1 + \eta_1 \cdot \delta C_c'' - \sigma_c'' \cdot \delta\varepsilon_1 - \sigma_1 \cdot \delta\varepsilon_c''] \\ & -a\lambda_c[\delta B_2 + \eta_c' \cdot \delta C_2 + \eta_2 \cdot \delta C_c' - \sigma_c' \cdot \delta\varepsilon_2 - \sigma_2 \cdot \delta\varepsilon_c'] \\ & -\lambda_c \delta B_3 - C_3 \cdot \delta\eta - \eta_c \cdot \delta C_3 - \eta_3 \cdot \delta C_c - \eta_1 \cdot \delta C_2 \\ & -\eta_2 \cdot \delta C_1 + \sigma_c \cdot \delta\varepsilon_3 + \sigma_3 \cdot \delta\varepsilon_c + \sigma_2 \cdot \delta\varepsilon_1 + \sigma_1 \cdot \delta\varepsilon_2 \end{aligned} \tag{40}$$

which entails the unknown postbuckling field u_3 as well as b. A procedure analogous to the one employed in Section 6.1 will show that (40) may be written:

$$\mathscr{E}_{11}(u_3, \delta u) = \mathscr{G}_1^1(\delta u) + a\lambda_c \mathscr{G}_1^2(\delta u) + \tfrac{1}{2}(a\lambda_c)^2 \mathscr{G}_1^3(\delta u) + b\lambda_c \mathscr{F}_1^2(\delta u) \tag{41}$$

Since we do not intend to establish and solve the third order problem (41) we do not determine the expressions for $\mathscr{G}_1^i(\delta u)$. To obtain an expression for the second order postbuckling constant b, the buckling problem (30) with $\delta u = u_3$ is subtracted from (41) with $\delta u = u_1$.

The buckling problem (29) with $\delta u = u_3$ is:

$$0 = \mathscr{E}_{11}(u_1, u_3) \tag{42}$$

and the third order problem (41) with $\delta u = u_1$ is:

$$\mathscr{E}_{11}(u_3, u_1) = \mathscr{G}_1^1(u_1) + a\lambda_c \mathscr{G}_1^2(u_1) + \tfrac{1}{2}(a\lambda_c)^2 \mathscr{G}_1^3(u_1) + b\lambda_c \mathscr{F}_1^2(u_1) \tag{43}$$

After subtraction of (42) from (43) the expression for $b\lambda_c$ is easily found to be :

$$b\lambda_c = -\frac{\mathscr{G}_1^1(u_1) + a\lambda_c \mathscr{G}_1^2(u_1) + \tfrac{1}{2}(a\lambda_c)^2 \mathscr{G}_1^3(u_1)}{\mathscr{F}_1^2(u_1)} \tag{44}$$

In order to determine expressions for $\mathscr{G}_1^i$ we insert the operator expansions (21) and (22) into (40) and compare with (41) with the result that:

$$\begin{aligned} \mathscr{G}_1^1(u_1) = {} & -\lambda_c[2\mathscr{B}_{12}(u_2, u_1) + 6\mathscr{B}_{112}(u_c, u_2, u_1) + \mathscr{B}_4(u_1)] \\ & -\eta_c \cdot [2\mathscr{C}_{12}(u_2, u_1) + 6\mathscr{C}_{112}(u_c, u_2, u_1) + \mathscr{C}_4(u_1)] \\ & -\eta_1 \cdot [2\mathscr{C}_{11}(u_1, u_2) + 4\mathscr{C}_{111}(u_c, u_1, u_2) + \tfrac{4}{3}\mathscr{C}_3(u_1) \\ & + 6\mathscr{C}_{112}(u_1, u_2, u_c) + 4\mathscr{C}_{13}(u_c, u_1)] - \eta_2 \cdot [\mathscr{C}_2(u_1 + 2\mathscr{C}_{12}(u_c, u_1) + 3\mathscr{C}_{22}(u_c, u_1)] \\ & + 2\sigma_1 \cdot \mathscr{L}_{11}(u_1, u_2) + \sigma_2 \cdot \mathscr{L}_2(u_1) \end{aligned} \tag{45}$$

$$
\begin{aligned}
\mathscr{G}_1^2(u_1) = {} & -\mathscr{B}_{11}(u_1, u_2) + \mathscr{B}_3(u_1) + 2\mathscr{B}_{111}(u_1, u_c, u_2) \\
& -3\mathscr{B}_{13}(u_c, u_1) + 3\mathscr{B}_{112}(u_1, u_2, u_c) \\
& -\lambda_c[2\mathscr{B}_{111}(u_1, u_c', u_2) + 3\mathscr{B}_{13}(u_c', u_1) + 6\mathscr{B}_{1111}(u_1, u_2, u_c, u_c')] \\
& -\eta_c' \cdot [\mathscr{C}_{11}(u_1, u_2) + \mathscr{C}_3(u_1) + 2\mathscr{C}_{111}(u_1, u_c, u_2) + 3\mathscr{C}_{13}(u_c, u_1) + 3\mathscr{C}_{112}(u_1, u_2, u_c)] \\
& -\eta_c \cdot [2\mathscr{C}_{111}(u_1, u_c', u_2) + 3\mathscr{C}_{13}(u_c', u_1) + 6\mathscr{C}_{1111}(u_1, u_2, u_c, u_c')] \\
& -\eta_1 \cdot [\mathscr{C}_{11}(u_c', u_2) + 3\mathscr{C}_{12}(u_c', u_1) + 2\mathscr{C}_{111}(u_c', u_c, u_2) \\
& +3\mathscr{C}_{112}(u_c', u_2, u_c) + 9\mathscr{C}_{112}(u_c, u_c', u_1)] \\
& -\eta_2 \cdot [\mathscr{C}_{11}(u_1, u_c') + 2\mathscr{C}_{111}(u_c, u_c', u_1) + 3\mathscr{C}_{112}(u_1, u_c', u_c)] \\
& +\sigma_c' \cdot \mathscr{L}_{11}(u_1, u_2) + \sigma_2 \cdot \mathscr{L}_{11}(u_1, u_c') + \sigma_1 \cdot \mathscr{L}_{11}(u_2, u_c')
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
\mathscr{G}_1^3(u_1) = {} & -2\mathscr{B}_{12}(u_c', u_1) + 6\mathscr{B}_{112}(u_c, u_c', u_1) \\
& -\lambda_c[2\mathscr{B}_{12}(u_c'', u_1) + 6\mathscr{B}_{22}(u_c', u_1) + 6\mathscr{B}_{112}(u_c, u_c'', u_1)] \\
& -\eta_c'' \cdot [\mathscr{C}_2(u_1) + 2\mathscr{C}_{12}(u_c, u_1) + 3\mathscr{C}_{22}(u_c, u_1)] \\
& -\eta_c' \cdot [2\mathscr{C}_{12}(u_c', u_1) + 6\mathscr{C}_{112}(u_c, u_c', u_1)] \\
& -\eta_c \cdot [2\mathscr{C}_{12}(u_c'', u_1) + 6\mathscr{C}_{22}(u_c', u_1) + 6\mathscr{C}_{112}(u_c, u_c'', u_1)] \\
& -\eta_1 \cdot [2\mathscr{C}_{11}(u_c'', u_1) + 4\mathscr{C}_{12}(u_1, u_c') + 4\mathscr{C}_{111}(u_c'', u_c, u_1) \\
& +12\mathscr{C}_{112}(u_c, u_1, u_c') + 6\mathscr{C}_{112}(u_c'', u_1, u_c)] + \sigma_c'' \cdot \mathscr{L}_2(u_1) + 2\sigma_1 \cdot \mathscr{L}_{11}(u_1, u_c'')
\end{aligned}
\tag{47}
$$

and the denominator $\mathscr{F}_1^2(u_1)$ is given by (36).

6.3. *The orthogonality condition*

The left hand side of (34) is identical to the eigenvalue problem (30) and is therefore-singular. In general, the complete solution (34) contains parts of the u_1-field and takes the form:

$$u_2 = u_2^{partic} + k_2 u_1 \tag{48}$$

where u_2^{partic} is a particular solution to (34) and k_2 is an arbitrary constant. To determine k_2, and through that u_2, an orthogonality condition between the buckling field and the higher order fields, i.e. u_2, $u_3, \ldots$, is introduced. As regards the orthogonality condition, which in the following is given by the bilinear operator Q_{11}, the conditions below must apply, see e.g. Budiansky (1974) or Fitch (1968):

$$Q_{11}(u_1, u_1) \neq 0 \quad Q_{11}(u_1, u_j) = 0, \quad j = 2, 3, \ldots \tag{49}$$

The amount of participation of the buckling mode u_1 in an arbitrary displacement field is then given as:

$$\xi = \frac{Q_{11}(u - u_0, u_1)}{Q_{11}(u_1, u_1)} \tag{50}$$

where

$$u = u_0 + \xi u_1 + \xi^2 u_2 + \ldots \tag{51}$$

In the general case, the postbuckling coefficient b, which depends on u_2, will depend on the orthogonality condition (49) imposed on u_2. Such an ambiguity in the determination of b apparently implies that the predictions of postbuckling behavior by means of b becomes inconsistent. However, the orthogonality condition is closely connected to the definition of the perturbation parameter ξ, see (50), and this fact is therefore important to bear in mind when the results of a postbuckling analysis are inspected in that the physical interpretation of the buckling mode amplitude ξ depends on the choice of Q_{11}.

6.3.1. *Symmetric case*: $a = 0$. The denominator of b is independent of u_2, while its numerator $b^N_{a=0}$ is given by:

$$b^N_{a=0}\lambda_c = -\mathscr{G}^1_1(u_1) \tag{52}$$

Let $\mathscr{G}^{partic}$ denote $\mathscr{G}^1_1(u_1)$ with u_2^{partic} substituted for u_2 and insert (48) in (52) to get:

$$b^N_{a=0} = -\mathscr{G}^{partic} + 2k_2[\lambda_c[\mathscr{B}_3(u_1) + 3\mathscr{B}_{13}(u_c, u_1)] + \eta_1 \cdot [\tfrac{3}{2}\mathscr{C}_2(u_1) + 3\mathscr{C}_{12}(u_c, u_1) + \tfrac{9}{2}\mathscr{C}_{22}(u_c, u_1)]$$
$$+ \eta_c \cdot [\mathscr{C}_3(u_1) + 3\mathscr{C}_{13}(u_c, u_1)] - \tfrac{3}{2}\sigma_1 \cdot \mathscr{L}_2(u_1)] \tag{53}$$

which, in view of (39), can be written:

$$b^N_{a=0} = -\mathscr{G}^{partic} - 2k_2\mathscr{F}^1_1(u_1) \tag{54}$$

When the numerator $\mathscr{F}^1_1(u_1)$ of a vanishes (54) is reduced to:

$$b^N_{a=0} = -\mathscr{G}^{partic} \tag{55}$$

which is independent of k_2. Thus for $a = 0$, the second order postbuckling constant b is independent of the choice of orthogonality condition. When $a = 0$ the sole purposes of imposing an orthogonality condition are to render the left hand side of (34) non-singular and to fix u_2 such that it does not contain any contribution of the buckling mode u_1.

6.3.2. *General case*: $a \neq 0$. For $a \neq 0$ the value of the second order postbuckling constant b may be written:

$$b^N_{a\neq 0} = -\mathscr{G}^{partic} - 3k_2\mathscr{F}^1_1(u_1) \tag{56}$$

which depends on k_2. Clearly, in this case the value of b hinges on the choice of orthogonality condition.

6.3.3. *Choice of orthogonality condition.* Obviously, there are several meaningful choices of orthogonality condition. The one which seems the most natural to impose on the second order field u_2 is:

$$\sigma'_c \cdot \mathscr{L}_{11}(u_1, u_2) = 0 \tag{57}$$

because it is identical to the most commonly used orthogonality condition when the prebuckling path is linear. It may be observed that for the case of nonlinear prebuckling the condition depends on the prebuckling field at buckling, which is meaningful because the asymptotic expansion is performed at (λ_c, u_c).

7. QUADRATIC LOADING AND AUXILIARY CONDITIONS

In this section we present the above developed formulas for a common case, namely one which entails loading terms and auxiliary conditions that are quadratic, not quartic, in the displacements. These assumptions reduce the enormity of the previously derived formulas considerably. Moreover, if the first order postbuckling constant a does not vanish, there is usually no reason to go beyond further and solve the first postbuckling problem in order to determine the second order postbuckling constant b.

7.1. *Prebuckling*

Here, (28) becomes:

$$H[\mathscr{L}_1(u_0)+\tfrac{1}{2}\mathscr{L}_2(u_0)]\cdot[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_0)] = \lambda[\mathscr{B}_1(\delta u)+\mathscr{B}_{11}(\delta u,u_0)] + \delta\eta\cdot[\mathscr{C}_1(u_0)+\tfrac{1}{2}\mathscr{C}_2(u_0)]+\eta_0\cdot[\mathscr{C}_1(\delta u)+\mathscr{C}_{11}(\delta u,u_0)] \quad (58)$$

7.2. *Buckling*

The variational eqn (30) governing buckling simplifies to:

$$\begin{aligned} 0 = &-\lambda_c\mathscr{B}_{11}(\delta u,u_1)-\delta\eta\cdot[\mathscr{C}_1(u_1)+\mathscr{C}_{11}(u_1,u_c)]-\eta_c\cdot\mathscr{C}_{11}(\delta u,u_1) \\ &-\eta_1\cdot[\mathscr{C}_1(\delta u)+\mathscr{C}_{11}(\delta u,u_c)]+\sigma_c\cdot\mathscr{L}_{11}(\delta u,u_1) \\ &+H[\mathscr{L}_1(u_1)+\mathscr{L}_{11}(u_1,u_c)]\cdot[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_c)] \end{aligned} \quad (59)$$

and the expression (39) giving the first order postbuckling constant a is:

$$a\lambda_c = \frac{3}{2}\frac{\eta_1\cdot\mathscr{C}_2(u_1)-\sigma_1\cdot\mathscr{L}_2(u_1)}{\sigma_c'\cdot\mathscr{L}_2(u_1)+2\sigma_1\cdot\mathscr{L}_{11}(u_1,u_c')-\mathscr{B}_2(u_1)-\eta_c'\cdot\mathscr{C}_2(u_1)-2\eta_1\cdot\mathscr{C}_{11}(u_1,u_c')} \quad (60)$$

7.3. *Postbuckling of symmetric structures*: $a = 0$

When the first order postbuckling constant a vanishes the first order postbuckling problem (34) reduces to:

$$\begin{aligned} &-\lambda_c\mathscr{B}_{11}(\delta u,u_2)-\delta\eta\cdot[\mathscr{C}_1(u_2)+\mathscr{C}_{11}(u_c,u_2)]-\eta_c\cdot\mathscr{C}_{11}(\delta u,u_2) \\ &-\eta_2\cdot[\mathscr{C}_1(\delta u)+\mathscr{C}_{11}(\delta u,u_c)]+\sigma_c\cdot\mathscr{L}_{11}(\delta u,u_2) \\ &+[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_c)]\cdot H[\mathscr{L}_1(u_2)+\mathscr{L}_{11}(u_c,u_2)] \\ &= \delta\eta\cdot\tfrac{1}{2}\mathscr{C}_2(u_1)+\eta_1\cdot\mathscr{C}_{11}(\delta u,u_1)-\sigma_1\cdot\mathscr{L}_{11}(\delta u,u_1) \\ &-[\mathscr{L}_1(\delta u)+\mathscr{L}_{11}(\delta u,u_c)]\cdot H[\tfrac{1}{2}\mathscr{L}_2(u_1)] \end{aligned} \quad (61)$$

and the expression (44) for the second order postbuckling constant b is:

$$b\lambda_c = \frac{2\eta_1\cdot\mathscr{C}_{11}(u_1,u_2)+\eta_2\cdot\mathscr{C}_2(u_1)-2\sigma_1\cdot\mathscr{L}_{11}(u_1,u_2)-\sigma_2\cdot\mathscr{L}_2(u_1)}{\sigma_c'\cdot\mathscr{L}_2(u_1)+2\sigma_1\cdot\mathscr{L}_{11}(u_1,u_c')-\mathscr{B}_2(u_1)-\eta_c'\cdot\mathscr{C}_2(u_1)-2\eta_1\cdot\mathscr{C}_{11}(u_1,u_c')} \quad (62)$$

8. EXAMPLE: THE COMPLETE RING—FULL NONLINEAR THEORY

As an example of an application of our theory we present the complete ring under hydrostatic load, see Fig. 2. The behavior of this structure has been investigated by, among others, Sills and Budiansky (1978) and Budiansky (1974). In both these references, the analysis was based on a formulation in terms of the potential energy of the system, while our analysis takes the modified principle of virtual displacements as its point of departure. Among the assumptions in (Sills and Budiansky, 1978) and (Budiansky, 1974) are inextensionality, which they enforce through application of Lagrange multipliers in much the same way that we do it below, the main difference being that in our analysis we utilize the general formulas derived above.

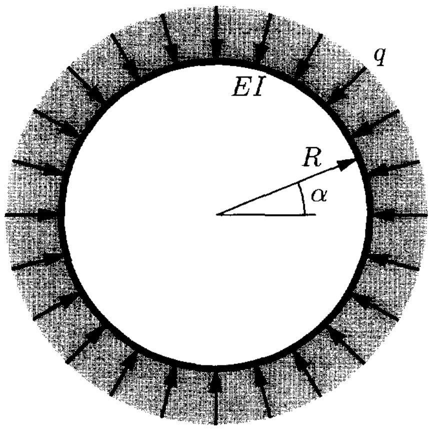

Fig. 2. The complete ring.

The load parameter λ is given by:

$$\lambda = \frac{\bar{q}R^3}{EI} \tag{63}$$

where $\bar{q}$ is the applied hydrostatic pressure, R is the radius of the ring, and EI is its bending stiffeness.

Although the beam theory is fully nonlinear the second order operator $\mathscr{L}_2$ of the strain-displacement relation vanishes in that:

$$\mathscr{L}_1(u) = \kappa = \frac{\mathrm{d}\theta}{\mathrm{d}\alpha} \quad \mathscr{L}_2(u) = 0 \tag{64}$$

where the bending strain κ is the only nonvanishing strain component, θ is the rotation of the beam axis, and α is the sectorial angle. In addition to θ the displacement field u contains the nondimensional axial displacement component v and the nondimensional transverse displacement component w, which are equal to the physical quantities divided by R.

8.1. *Virtual work*

Here, the auxiliary conditions describe inextensibility and the connection between θ and v, w, respectively. Thus, the operator C of (4) is:

$$C(u) = \begin{bmatrix} w + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha} + \dfrac{1}{2}\left(\left(w + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}\right)^2 + \left(\dfrac{\mathrm{d}w}{\mathrm{d}\alpha} - v\right)^2\right) \\ \sin(\theta) + \dfrac{\mathrm{d}w}{\mathrm{d}\alpha} - v \end{bmatrix} = 0 \tag{65}$$

When we expand C to order 4, the following expressions are found:

$$\mathscr{C}_1(u) = \begin{bmatrix} w + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha} \\ \theta + \dfrac{\mathrm{d}w}{\mathrm{d}\alpha} - v \end{bmatrix} \quad \mathscr{C}_2(u) = \begin{bmatrix} \left(w + \dfrac{\mathrm{d}v}{\mathrm{d}\alpha}\right)^2 + \left(\dfrac{\mathrm{d}w}{\mathrm{d}\alpha} - v\right)^2 \\ 0 \end{bmatrix} \quad \mathscr{C}_3(u) = \begin{bmatrix} 0 \\ -\frac{1}{2}\theta^3 \end{bmatrix} \tag{66}$$

see (12c).

The load potential $B(u)$ for hydrostatic loading is, see e.g. (Sills and Budiansky, 1978):

$$B(u) = -\int_0^{2\pi}\left(w - v\frac{\mathrm{d}w}{\mathrm{d}\alpha} + \tfrac{1}{2}w^2 + \tfrac{1}{2}v^2\right)\mathrm{d}\alpha = -\int_0^{2\pi}\left(w - \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right)^2 + \tfrac{1}{2}w^2 + \frac{1}{2}\left(v - \frac{\mathrm{d}w}{\mathrm{d}\alpha}\right)^2\right)\mathrm{d}\alpha \tag{67}$$

and thus:

$$\mathscr{B}_1(u) = -\int_0^{2\pi} w\,\mathrm{d}\alpha$$

$$\mathscr{B}_2(u) = \int_0^{2\pi}\left(\left(\frac{\mathrm{d}w}{\mathrm{d}\alpha}\right)^2 - w^2 - \left(v - \frac{\mathrm{d}w}{\mathrm{d}\alpha}\right)^2\right)\mathrm{d}\alpha \tag{68}$$

8.2. *Prebuckling*

Utilize (28) to get:

$$\begin{aligned}
0 = &+\int_0^{2\pi}\frac{\mathrm{d}^2\theta_0}{\mathrm{d}\alpha^2}\delta\theta\,\mathrm{d}\alpha \\
&-\int_0^{2\pi}\lambda\left(\delta w + w_0\delta w + \frac{\mathrm{d}^2 w_0}{\mathrm{d}\alpha^2}\delta w + v_0\delta v - \frac{\mathrm{d}^2 w_0}{\mathrm{d}\alpha^2}\delta w + \frac{\mathrm{d}v_0}{\mathrm{d}\alpha}\delta w - \frac{\mathrm{d}w_0}{\mathrm{d}\alpha}\delta v\right)\mathrm{d}\alpha \\
&+\int_0^{2\pi}\delta\eta^1\left(w_0 + \frac{\mathrm{d}v_0}{\mathrm{d}\alpha} + \frac{1}{2}\left(\left(w_0 + \frac{\mathrm{d}v_0}{\mathrm{d}\alpha}\right)^2 + \left(\frac{\mathrm{d}w_0}{\mathrm{d}\alpha} - v_0\right)^2\right)\right)\mathrm{d}\alpha \\
&+\int_0^{2\pi}\left(\eta_0^1\delta w + \frac{\mathrm{d}\eta_0^1}{\mathrm{d}\alpha}\delta v + \eta_0^1 w_0\delta w + \eta_0^1\frac{\mathrm{d}v_0}{\mathrm{d}\alpha}\delta w - \frac{\mathrm{d}(\eta_0^1 w_0)}{\mathrm{d}\alpha}\delta v - \frac{\mathrm{d}(\eta_0^1\,\mathrm{d}v_0/\mathrm{d}\alpha)}{\mathrm{d}\alpha}\delta v\right. \\
&\left.-\frac{\mathrm{d}(\eta_0^1\,\mathrm{d}w_0/\mathrm{d}\alpha)}{\mathrm{d}\alpha}\delta w + \frac{\mathrm{d}(\eta_0^1 v_0)}{\mathrm{d}\alpha}\delta w - \eta_0^1\frac{\mathrm{d}w_0}{\mathrm{d}\alpha}\delta v + \eta_0^1 v_0\delta v\right)\mathrm{d}\alpha \\
&+\int_0^{2\pi}\delta\eta^2\left(\frac{\mathrm{d}w_0}{\mathrm{d}\alpha} - v_0 + \theta_0 - \tfrac{1}{6}\theta_0^3\right)\mathrm{d}\alpha + \int_0^{2\pi}\left(-\frac{\mathrm{d}\eta_0^2}{\mathrm{d}\alpha}\delta w - \eta_0^2\delta v + \eta_0^2\delta\theta\right)\mathrm{d}\alpha
\end{aligned} \tag{69}$$

Here:

$$\eta_0 = (\eta_0^1, \eta_0^2) \tag{70}$$

where η^1 and η^2 are associated with the condition of inextensibility and with the connection between rotation and displacements, respectively, see (65) and (66).

It is easily seen that the solution to (69) is:

$$w_0 = v_0 = \theta_0 = \eta_0^2 = 0 \quad \eta_0^1 = \lambda \tag{70}$$

8.3. *Buckling*

Equation (30) provides:

$$\begin{aligned}
0 = &+\int_0^{2\pi}\frac{\mathrm{d}^2\theta_1}{\mathrm{d}\alpha^2}\delta\theta\,\mathrm{d}\alpha \\
&+\int_0^{2\pi}\lambda_c\left(w_1\delta w - \frac{\mathrm{d}^2 v_1}{\mathrm{d}\alpha^2}\delta v - \frac{\mathrm{d}w_1}{\mathrm{d}\alpha}\delta v + \frac{\mathrm{d}v_1}{\mathrm{d}\alpha}\delta w - \frac{\mathrm{d}^2 w_1}{\mathrm{d}\alpha^2}\delta w + v_1\delta v + \frac{\mathrm{d}v_1}{\mathrm{d}\alpha}\delta w\right)\mathrm{d}\alpha
\end{aligned}$$

$$+\int_0^{2\pi}\left(\eta_1^1\delta w-\frac{\mathrm{d}\eta_1^1}{\mathrm{d}\alpha}\delta v\right)\mathrm{d}\alpha+\int_0^{2\pi}\delta\eta^1\left(w_1+\frac{\mathrm{d}v_1}{\mathrm{d}\alpha}\right)\mathrm{d}\alpha$$

$$+\int_0^{2\pi}\left(\eta_1^2\delta\theta-\frac{\mathrm{d}\eta_1^2}{\mathrm{d}\alpha}\delta w-\eta_1^2\delta v\right)\mathrm{d}\alpha+\int_0^{2\pi}\delta\eta^2\left(\theta_1+\frac{\mathrm{d}w_1}{\mathrm{d}\alpha}-v_1\right)\mathrm{d}\alpha$$

$$-\int_0^{2\pi}\lambda_c\left(w_1\delta w+\frac{\mathrm{d}^2w_1}{\mathrm{d}\alpha^2}\delta w+v_1\delta v-\frac{\mathrm{d}^2w_1}{\mathrm{d}\alpha^2}\delta w-\frac{\mathrm{d}w_1}{\mathrm{d}\alpha}\delta v+\frac{\mathrm{d}v_1}{\mathrm{d}\alpha}\delta w\right)\mathrm{d}\alpha \tag{72}$$

Excluding rigid body displacements the solution to (72) is:

$$w_1=\cos(2\alpha)\quad v_1=-\tfrac{1}{2}\sin(2\alpha)\quad \theta_1=\tfrac{3}{2}\sin(2\alpha)\quad \eta_1^1=3\cos(2\alpha)\quad \eta_1^2=6\sin(2\alpha) \tag{73}$$

where the buckling mode is normalized such that the amplitude of the transverse displacement component is 1, and the classical critical load is given by:

$$\lambda_c=3 \tag{74}$$

8.4. *Postbuckling*

It is easily verified that the ring must exhibit a symmetric postbuckling behavior. Thus, the first order postbuckling constant a vanishes:

$$a=0 \tag{75}$$

Therefore, we need the postbuckling field u_2 in order to determine the second order postbuckling constant b. For $a=0$ the first postbuckling problem given by (34) may provide the following variational equation for u_2:

$$-\int_0^{2\pi}\frac{\mathrm{d}^2\theta_2}{\mathrm{d}\alpha^2}\delta\theta\,\mathrm{d}\alpha$$

$$-\int_0^{2\pi}\lambda_c\left(w_2\delta w-\frac{\mathrm{d}^2v_2}{\mathrm{d}\alpha^2}\delta v-\frac{\mathrm{d}w_2}{\mathrm{d}\alpha}\delta v+\frac{\mathrm{d}v_2}{\mathrm{d}\alpha}\delta w-\frac{\mathrm{d}^2w_2}{\mathrm{d}\alpha^2}\delta w+v_2\delta v+\frac{\mathrm{d}v_2}{\mathrm{d}\alpha}\delta w\right)\mathrm{d}\alpha$$

$$-\int_0^{2\pi}\left(\eta_2^1\delta w-\frac{\mathrm{d}\eta_2^1}{\mathrm{d}\alpha}\delta v\right)\mathrm{d}\alpha-\int_0^{2\pi}\delta\eta^1\left(w_2+\frac{\mathrm{d}v_2}{\mathrm{d}\alpha}\right)\mathrm{d}\alpha$$

$$-\int_0^{2\pi}\left(\eta_2^2\delta\theta-\frac{\mathrm{d}\eta_2^2}{\mathrm{d}\alpha}\delta w-\eta_2^2\delta v\right)\mathrm{d}\alpha-\int_0^{2\pi}\delta\eta^2\left(\theta_2+\frac{\mathrm{d}w_2}{\mathrm{d}\alpha}-v_2\right)\mathrm{d}\alpha$$

$$+\int_0^{2\pi}\lambda_c\left(w_2\delta w+\frac{\mathrm{d}^2w_2}{\mathrm{d}\alpha^2}\delta w+v_2\delta v-\frac{\mathrm{d}^2w_2}{\mathrm{d}\alpha^2}\delta w-\frac{\mathrm{d}w_2}{\mathrm{d}\alpha}\delta v+\frac{\mathrm{d}v_2}{\mathrm{d}\alpha}\delta w\right)\mathrm{d}\alpha$$

$$=+\int_0^{2\pi}\tfrac{9}{16}(1-\cos(4\alpha))\delta\eta^1\,\mathrm{d}\alpha+\int_0^{2\pi}(\tfrac{9}{4}\sin(4\alpha)\delta v+9\cos(4\alpha)\delta w)\,\mathrm{d}\alpha \tag{76}$$

When the orthogonality condition (49b) and the condition of periodicity are exploited, and all rigid body terms are eliminated, we get the following solution to (76):

$$w_2 = -\tfrac{9}{16} \quad v_2 = \tfrac{9}{64}\sin(4\alpha) \quad \theta_2 = \tfrac{9}{64}\sin(4\alpha) \quad \eta_2^1 = -\tfrac{27}{16}\cos(4\alpha) \quad \eta_2^2 = \tfrac{9}{4}\sin(4\alpha) \tag{77}$$

Finally, after some computations, (62) yields:

$$b\lambda_c = \tfrac{81}{32} \tag{78}$$

8.5. *Comments*

If we had retained only one term in the expansion for $\sin(\theta)$ in (65), which has been done by e.g. Rehfield (1972), the prediction for the buckling load would remain the same, but the prediction of the postbuckling behavior would be in error in that the value of $b\lambda_c$ would be $-\frac{27}{32}$ instead of $+\frac{81}{32}$, which is due to Sills and Budiansky (1978). On the other hand, it is possible to show that the value of $b\lambda_c$ is unaltered whether or not higher order terms in $\sin(\theta)$ are included.

We have utilized the ring example to show the application of the—admittedly rather lengthy formulas of our theory—and to corroborate the finding of Sills and Budiansky (1978) that it may be important to choose a bending strain measure which is more accurate than the usual linear.

9. CONCLUSION

A set of equations for prebuckling, buckling and postbuckling of structures, which exhibit strong nonlinear behavior and are subjected to nonlinear loads as well as nonlinear auxiliary conditions are derived in the main body of this paper. Although the full formulas are rather lengthy, in most applications they simplify considerably. The main advantage of the theory lies in the fact that it encompasses most relevant cases, and therefore major efforts in deriving formulas for more specialized cases can be avoided.

The finding of Sills and Budiansky (1978) that in order to describe postbuckling behavior, in particular of symmetric structures such as the complete ring, it is important to retain load terms of high order is corroborated by the present analysis.

REFERENCES

Belytschko, T., Stolarski, H., Liu, W., Carpenter, N. and Ong, J.-J. (1985). Stress projection for membrane and shear locking in shell finite elements. *Computer Meth. in Appl. Mech. and Engng* **51,** 221–258.

Budiansky, B. (1974). Theory of buckling and post-buckling behavior of elastic structures. *Advances in Appl. Mech.* **14,** 1–65.

Budiansky, B. and Hutchinson, J. W. (1964). Dynamic buckling of imperfection-sensitive structures. In *Proc. 11th Int. Congress on Applied mechanics*, (ed. H. Görtler) pp. 636–651, Springer-Verlag, Berlin.

Byskov, E. (1989a). Selected buckling problems. Dissertation for the degree of Dr Techn., The Technical University of Denmark, Lyngby, Denmark.

Byskov, E. (1989b). Smooth postbuckling stresses by a modified finite element method. *Int. J. Num. Meth. Engng* **28,** 2877–2888.

Fitch, J. (1968). The buckling and post-buckling behavior of spherical caps under concentrated load. *Int. J. Solids Structures* **4,** 421–466.

Hutchinson, J. W. (1974). Unpublished notes.

Koiter, W. T. (1945). On the stability of elastic equilibrium. (Dissertation, Delft University), H. J. Paris, Amsterdam. (in Dutch); also *NASA TT-F*10, 833, (1967); also Air Force Flight Dynamics Laboratory, *AFFDL-TR*-70-25, (1970).

Mau, S. and El-Mabsout, M. (1989). Inelastic buckling of reinforcing bars. *J. Engng Mech.* **115,** 1–17.

Noor, A. and Peters, J. (1981). Mixed models and reduced/selective integration displacement models for nonlinear analysis of curved beams. *Int. J. Num. Meth. Engng* **17,** 615–631.

Rehfield, L. (1972). Initial postbuckling of circular rings under pressure loads. *AIAA Journal* **10,** 1358–1359.

Sills, L. and Budiansky, B. (1978). Postbuckling ring analysis. *Trans. ASME* **45,** 208–210.

Stolarski, H. and Belytschko, T. (1983). Shear and membrane locking in curved C^0 elements. *Computer Meth. in Appl. Mech. and Engng* **41,** 279–296.

APPENDIX A

Operator rules

In the following we introduce a number of operators $\mathscr{P}_N(u)$, $N \in \{1, 2, 3, 4\}$, and give rules for differentiation etc. The general operator of degree 4 is written $\mathscr{P}_{ijkl}(u_a, u_b, u_c, u_d)$, where $i, j, k, l \in \{0, 1, 2, 3, 4\}$ denotes the order of u_a, u_b, u_c or u_d, respectively. The sum $i+j+k+l \leqslant 4$ denotes the order of the operator. If an index, e.g. l, is equal to zero that index may be left out and the operator can be written as an operator of a lower degree. As an example:

$$\mathscr{P}_{1110}(u_a, u_b, u_c, u_d) = \mathscr{P}_{111}(u_a, u_b, u_c) \tag{79}$$

Among the other operators of degree 3 are $\mathscr{P}_{12}(u_a, u_b)$ and $\mathscr{P}_3(u_a)$.
In the following, indices that are equal to 0 are ignored.

A.1. Symmetry properties. For any two indices, e.g. i and j, of an operator the following symmetry properties apply:

$$\mathscr{P}_{ij}(u_a, u_b) = \mathscr{P}_{ji}(u_b, u_a) \tag{80}$$

where the other indices have been left out. As an example of an operator of degree 4:

$$\mathscr{P}_{112}(u_a, u_b, u_c) = \mathscr{P}_{112}(u_b, u_a, u_c) = \mathscr{P}_{121}(u_a, u_c, u_b) = \mathscr{P}_{121}(u_b, u_c, u_a) = \mathscr{P}_{211}(u_c, u_a, u_b) = \mathscr{P}_{211}(u_c, u_b, u_c). \tag{81}$$

A.2. Multiplication by scalars. For scalars, e.g. s and t the rules for multiplication are:

$$\mathscr{P}_{ij}(su_a, tu_a) = s^i t^j \mathscr{P}_{i+j}(u_a). \tag{82}$$

A.3. Addition. The operators obey the following rules for addition:

$$\begin{aligned}
\mathscr{P}_2(u_a+u_b) &= \mathscr{P}_2(u_a) + 2\mathscr{P}_{11}(u_a, u_b) + \mathscr{P}_2(u_b)\\
\mathscr{P}_3(u_a+u_b) &= \mathscr{P}_3(u_a) + 3\mathscr{P}_{12}(u_a, u_b) + 3\mathscr{P}_{12}(u_b, u_a) + \mathscr{P}_3(u_b)\\
\mathscr{P}_4(u_a+u_b) &= \mathscr{P}_4(u_a) + 4\mathscr{P}_{13}(u_a, u_b) + 6\mathscr{P}_{22}(u_a, u_b) + 4\mathscr{P}_{13}(u_b, u_a) + \mathscr{P}_4(u_b)
\end{aligned} \tag{83}$$

A.4 Differentiation. Differentiation is performed according to the rules below:

$$\begin{aligned}
\mathscr{P}'_{ijkl}(u_a, u_b, u_c, u_d) = &+ i\mathscr{P}_{1(i-1)jkl}(u'_a, u_a, u_b, u_c, u_d) + j\mathscr{P}_{1i(j-1)kl}(u'_b, u_a, u_b, u_c, u_d) + k\mathscr{P}_{1ij(k-1)l}(u'_c, u_a, u_b, u_c, u_d)\\
&+ l\mathscr{P}_{1ijk(l-1)}(u'_d, u_a, u_b, u_c, u_d)
\end{aligned} \tag{84}$$

Observe that at least one of the 5 indices in each term must vanish, as the sum of the indices does not exceed 4.

APPENDIX B

Perturbation constants, p_c, p_1, p_2 and p_3

To determine p_1, p_2 and p_3 in the perturbation expansion for p, see (8), the expression for u near the point of bifurcation (17) is inserted into (10). This provides the following expressions (85)–(88):

$$\begin{aligned}
\mathscr{P}_1(u) = \mathscr{P}_1(u_c) &+ \xi[a\lambda_c\mathscr{P}_1(u'_c) + \mathscr{P}_1(u_1)] + \xi^2[b\lambda_c\mathscr{P}_1(u'_c) + \tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_1(u''_c) + \mathscr{P}_1(u_2)]\\
&+ \xi^3[c\lambda_c\mathscr{P}_1(u'_c) + ab\lambda_c^2\mathscr{P}_1(u''_c) + \tfrac{1}{6}(a\lambda_c)^3\mathscr{P}_1(u'''_c) + \mathscr{P}_1(u_3)] + O(\xi^4)
\end{aligned} \tag{85}$$

$$\begin{aligned}
\tfrac{1}{2}\mathscr{P}_2(u) = \tfrac{1}{2}\mathscr{P}_2(u_c) &+ \xi[a\lambda_c\mathscr{P}_{11}(u_c, u'_c) + \mathscr{P}_{11}(u_c, u_1)]\\
&+ \xi^2[\tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_2(u'_c) + a\lambda_c\mathscr{P}_{11}(u'_c, u_1) + \tfrac{1}{2}\mathscr{P}_2(u_1)\\
&+ b\lambda_c\mathscr{P}_{11}(u_c, u'_c) + \tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_{11}(u_c, u''_c) + \mathscr{P}_{11}(u_c, u_2)]\\
&+ \xi^3[ab\lambda_c^2\mathscr{P}_2(u'_c) + \tfrac{1}{2}(a\lambda_c)^3\mathscr{P}_{11}(u'_c, u''_c) + a\lambda_c\mathscr{P}_{11}(u'_c, u_2)\\
&+ b\lambda_c\mathscr{P}_{11}(u'_c, u_1) + \tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_{11}(u''_c, u_1) + \mathscr{P}_{11}(u_1, u_2)\\
&+ c\lambda_c\mathscr{P}_{11}(u_c, u'_c) + ab\lambda_c^2\mathscr{P}_{11}(u_c, u''_c) + \tfrac{1}{6}(a\lambda_c)^3\mathscr{P}_{11}(u_c, u'''_c) + \mathscr{P}_{11}(u_c, u_3)]\\
&+ O(\xi^4)
\end{aligned} \tag{86}$$

$$\begin{aligned}
\tfrac{1}{3}\mathscr{P}_3(u) = \tfrac{1}{3}\mathscr{P}_3(u_c) &+ \xi[a\lambda_c\mathscr{P}_{12}(u'_c, u_c) + \mathscr{P}_{12}(u_1, u_c)]\\
&+ \xi^2[(a\lambda_c)^2\mathscr{P}_{12}(u_c, u'_c) + 2a\lambda_c\mathscr{P}_{111}(u_c, u'_c, u_1) + \mathscr{P}_{12}(u_c, u_1)\\
&+ b\lambda_c\mathscr{P}_{12}(u'_c, u_c) + \tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_{12}(u''_c, u_c) + \mathscr{P}_{12}(u_2, u_c)]\\
&+ \xi^3[2ab\lambda_c^2\mathscr{P}_{12}(u_c, u'_c) + (a\lambda_c)^3\mathscr{P}_{111}(u_c, u'_c, u''_c)\\
&+ 2a\lambda_c\mathscr{P}_{111}(u_c, u'_c, u_2) + 2b\lambda_c\mathscr{P}_{111}(u_c, u'_c, u_1)
\end{aligned}$$

$$+(a\lambda_c)^2\mathscr{P}_{111}(u_c,u_c'',u_1)+2\mathscr{P}_{111}(u_c,u_1,u_2)+c\lambda_c\mathscr{P}_{12}(u_c',u_c)+ab\lambda_c^2\mathscr{P}_{12}(u_c'',u_c)$$
$$+\tfrac{1}{6}(a\lambda_c)^3\mathscr{P}_{12}(u_c''',u_c)+\mathscr{P}_{12}(u_3,u_c)+\tfrac{1}{3}(a\lambda_c)^3\mathscr{P}_3(u_c')+\tfrac{1}{3}\mathscr{P}_3(u_1)$$
$$+(a\lambda_c)^2\mathscr{P}_{12}(u_1,u_c')+a\lambda_c\mathscr{P}_{12}(u_c',u_1)]+O(\xi^4) \tag{87}$$

$$\tfrac{1}{4}\mathscr{P}_4(u)=\tfrac{1}{4}\mathscr{P}_4(u_c)+\xi[a\lambda_c\mathscr{P}_{13}(u_c',u_c)+\mathscr{P}_{13}(u_1,u_c)]$$
$$+\xi^2[\tfrac{3}{2}(a\lambda_c)^2\mathscr{P}_{22}(u_c,u_c')+3a\lambda_c\mathscr{P}_{112}(u_c',u_1,u_c)+\tfrac{3}{2}\mathscr{P}_{22}(u_c,u_1)$$
$$+b\lambda_c\mathscr{P}_{13}(u_c',u_c)+\tfrac{1}{2}(a\lambda_c)^2\mathscr{P}_{13}(u_c'',u_c)+\mathscr{P}_{13}(u_2,u_c)]$$
$$+\xi^3[3ab\lambda_c^2\mathscr{P}_{22}(u_c',u_c)+\tfrac{3}{2}(a\lambda_c)^3\mathscr{P}_{112}(u_c',u_c'',u_c)$$
$$+3a\lambda_c\mathscr{P}_{112}(u_c',u_2,u_c)+3b\lambda_c\mathscr{P}_{112}(u_c',u_1,u_c)$$
$$+\tfrac{3}{2}(a\lambda_c)^2\mathscr{P}_{112}(u_c'',u_1,u_c)+3\mathscr{P}_{112}(u_1,u_2,u_c)+c\lambda_c\mathscr{P}_{13}(u_c',u_c)+ab\lambda_c^2\mathscr{P}_{13}(u_c'',u_c)$$
$$+\tfrac{1}{6}(a\lambda_c)^3\mathscr{P}_{13}(u''',u_c)+\mathscr{P}_{13}(u_3,u_c)+(a\lambda_c)^3\mathscr{P}_{13}(u_c,u_c')+\mathscr{P}_{13}(u_c,u_1)$$
$$+3(a\lambda_c)^2\mathscr{P}_{112}(u_c,u_1,u_c')+3a\lambda_c\mathscr{P}_{112}(u_c,u_c',u_1)]+O(\xi^4) \tag{88}$$

By comparing terms with ξ of the same order in the Taylor expansion of p from (15), the operator expansion of p (85)–(88) and the fundamental assumptions (8)–(10) the following expressions for p_1, p_2 and p_3 are obtained:

$$\begin{aligned}p_1 = {} & -a\lambda_c p_c'+a\lambda_c[\mathscr{P}_1(u_c')+\mathscr{P}_{11}(u_c,u_c')+\mathscr{P}_{12}(u_c',u_c)+\mathscr{P}_{13}(u_c',u_c)]\\ & +\mathscr{P}_1(u_1)+\mathscr{P}_{11}(u_1,u_c)+\mathscr{P}_{12}(u_1,u_c)+\mathscr{P}_{13}(u_1,u_c)\end{aligned} \tag{89}$$

$$\begin{aligned}p_2 = {} & -b\lambda_c p_c'-\tfrac{1}{2}a^2\lambda_c^2 p_c''+a\lambda_c[\mathscr{P}_{11}(u_c',u_1)+2\mathscr{P}_{111}(u_c,u_c',u_1)+3\mathscr{P}_{112}(u_c',u_1,u_c)]\\ & +b\lambda_c[\mathscr{P}_1(u_c')+\mathscr{P}_{11}(u_c,u_c')+\mathscr{P}_{12}(u_c',u_c)+\mathscr{P}_{13}(u_c',u_c)]\\ & +\tfrac{1}{2}(a\lambda_c)^2[+\mathscr{P}_1(u_c'')+\mathscr{P}_{11}(u_c,u_c'')+\mathscr{P}_2(u_c')+2\mathscr{P}_{12}(u_c,u_c')\\ & +\mathscr{P}_{12}(u_c'',u_c)+3\mathscr{P}_{22}(u_c,u_c')+\mathscr{P}_{13}(u_c'',u_c)]\\ & +\mathscr{P}_1(u_2)+\tfrac{1}{2}\mathscr{P}_2(u_1)+\mathscr{P}_{11}(u_c,u_2)+\mathscr{P}_{12}(u_c,u_1)\\ & +\mathscr{P}_{12}(u_2,u_c)+\tfrac{3}{2}\mathscr{P}_{22}(u_c,u_1)+\mathscr{P}_{13}(u_2,u_c)\end{aligned} \tag{90}$$

and

$$\begin{aligned}p_3 = {} & -c\lambda_c p_c'-ab\lambda_c^2 p_c''-\tfrac{1}{6}a^3\lambda_c^3 p_c'''\\ & +a\lambda_c[+\mathscr{P}_{11}(u_c',u_2)+2\mathscr{P}_{111}(u_c,u_c',u_2)+\mathscr{P}_{12}(u_c',u_1)\\ & +3\mathscr{P}_{112}(u_c',u_2,u_c)+3\mathscr{P}_{112}(u_c,u_c',u_1)]\\ & +b\lambda_c[\mathscr{P}_{11}(u_c',u_1)+2\mathscr{P}_{111}(u_c,u_c',u_1)+3\mathscr{P}_{112}(u_c',u_1,u_c)]\\ & +c\lambda_c[\mathscr{P}_1(u_c')+\mathscr{P}_{11}(u_c,u_c')+\mathscr{P}_{12}(u_c',u_c)+\mathscr{P}_{13}(u_c',u_c)]\\ & +\tfrac{1}{2}(a\lambda_c)^2[+\mathscr{P}_{11}(u_c'',u_1)+2\mathscr{P}_{111}(u_c,u_c'',u_1)+2\mathscr{P}_{12}(u_1,u_c')\\ & +3\mathscr{P}_{112}(u_c'',u_1,u_c)+6\mathscr{P}_{112}(u_c,u_1,u_c')]\\ & +ab\lambda_c^2[+\mathscr{P}_1(u_c'')+\mathscr{P}_2(u_c')+\mathscr{P}_{11}(u_c,u_c'')+2\mathscr{P}_{12}(u_c,u_c')\\ & +\mathscr{P}_{12}(u_c'',u_c)+3\mathscr{P}_{22}(u_c',u_c)+\mathscr{P}_{13}(u_c'',u_c)]\\ & +\tfrac{1}{6}(a\lambda_c)^3[+\mathscr{P}_1(u_c''')+3\mathscr{P}_{11}(u_c',u_c'')+\mathscr{P}_{11}(u_c,u_c''')\\ & +6\mathscr{P}_{111}(u_c,u_c',u_c'')+\mathscr{P}_{12}(u_c''',u_c)+2\mathscr{P}_3(u_c')\\ & +9\mathscr{P}_{112}(u_c',u_c'',u_c)+\mathscr{P}_{13}(u_c''',u_c)+6\mathscr{P}_{13}(u_c,u_c')]\\ & +\mathscr{P}_1(u_3)+\mathscr{P}_{11}(u_1,u_2)+\mathscr{P}_{11}(u_c,u_3)+2\mathscr{P}_{111}(u_c,u_1,u_2)\\ & +\mathscr{P}_{12}(u_3,u_c)+\tfrac{1}{3}\mathscr{P}_3(u_1)+3\mathscr{P}_{112}(u_1,u_2,u_c)+\mathscr{P}_{13}(u_3,u_c)+\mathscr{P}_{13}(u_c,u_1)\end{aligned} \tag{91}$$

When we observe that:

$$\begin{aligned}p_c &= \mathscr{P}_1(u_c)+\tfrac{1}{2}\mathscr{P}_2(u_c)+\tfrac{1}{3}\mathscr{P}_3(u_c)+\tfrac{1}{4}\mathscr{P}_4(u_c)\\ p_c' &= \mathscr{P}_1(u_c')+\mathscr{P}_{11}(u_c,u_c')+\mathscr{P}_{12}(u_c',u_c)+\mathscr{P}_{13}(u_c',u_c)\\ p_c'' &= \mathscr{P}_1(u_c'')+\mathscr{P}_{11}(u_c,u_c'')+\mathscr{P}_2(u_c')+2\mathscr{P}_{12}(u_c,u_c')\\ &\quad +\mathscr{P}_{12}(u_c'',u_c)+3\mathscr{P}_{22}(u_c,u_c')+\mathscr{P}_{13}(u_c'',u_c)\\ p_c''' &= \mathscr{P}_1(u_c''')+3\mathscr{P}_{11}(u_c',u_c'')+\mathscr{P}_{11}(u_c,u_c''')\\ &\quad +6\mathscr{P}_{111}(u_c,u_c',u_c'')+\mathscr{P}_{12}(u_c''',u_c)+2\mathscr{P}_3(u_c')\\ &\quad +9\mathscr{P}_{112}(u_c',u_c'',u_c)+\mathscr{P}_{13}(u_c''',u_c)+6\mathscr{P}_{13}(u_c,u_c')\end{aligned} \tag{92}$$

the expressions for p_1, p_2 and p_3 may be reduced to (21).

APPENDIX C

The perturbation constants δp_c, δp_1, δp_2, and δp_3

The variations δp_1, δp_2 and δp_3 are determined in the same fashion as formulas for p_i were derived in Appendix B. The expression for u, see (17), is introduced in (11), which furnishes the identity:

$$\mathscr{P}_1(\delta u) = \mathscr{P}_1(\delta u) \tag{93}$$

and

$$\begin{aligned}\mathscr{P}_{11}(\delta u, u) = {} & \mathscr{P}_{11}(\delta u, u_c) + \xi[a\lambda_c \mathscr{P}_{11}(\delta u, u_c') + \mathscr{P}_{11}(\delta u, u_1)] \\ & + \xi^2[b\lambda_c \mathscr{P}_{11}(\delta u, u_c') + \tfrac{1}{2}(a\lambda_c)^2 \mathscr{P}_{11}(\delta u, u_c'') + \mathscr{P}_{11}(\delta u, u_2)] \\ & + \xi^3[c\lambda_c \mathscr{P}_{11}(\delta u, u_c') + ab\lambda_c^2 \mathscr{P}_{11}(\delta u, u_c'') \\ & + \tfrac{1}{6}(a\lambda_c)^3 \mathscr{P}_{11}(\delta u, u_c''') + \mathscr{P}_{11}(\delta u, u_3)] + O(\xi^4)\end{aligned} \tag{94}$$

$$\begin{aligned}\mathscr{P}_{12}(\delta u, u) = {} & \mathscr{P}_{12}(\delta u, u_c) + \xi[2a\lambda_c \mathscr{P}_{111}(\delta u, u_c, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_1)] \\ & + \xi^2[(a\lambda_c)^2 \mathscr{P}_{12}(\delta u, u_c') + 2a\lambda_c \mathscr{P}_{111}(\delta u, u_c', u_1) + \mathscr{P}_{12}(\delta u, u_1) \\ & + 2b\lambda_c \mathscr{P}_{111}(\delta u, u_c, u_c') + (a\lambda_c)^2 \mathscr{P}_{111}(\delta u, u_c, u_c'') + 2\mathscr{P}_{111}(\delta u, u_c, u_2)] \\ & + \xi^3[2ab\lambda_c^2 \mathscr{P}_{12}(\delta u, u_c') + (a\lambda_c)^3 \mathscr{P}_{111}(\delta u, u_c', u_c'') \\ & + 2a\lambda_c \mathscr{P}_{111}(\delta u, u_c', u_2) + 2b\lambda_c \mathscr{P}_{111}(\delta u, u_c', u_1) \\ & + (a\lambda_c)^2 \mathscr{P}_{111}(\delta u, u_c'', u_1) + 2\mathscr{P}_{111}(\delta u, u_1, u_2) \\ & + 2c\lambda_c \mathscr{P}_{111}(\delta u, u_c, u_c') + 2ab\lambda_c^2 \mathscr{P}_{111}(\delta u, u_c, u_c'') \\ & + \tfrac{1}{3}(a\lambda_c)^3 \mathscr{P}_{111}(\delta u, u_c, u_c''') + 2\mathscr{P}_{111}(\delta u, u_c, u_3)] + O(\xi^4)\end{aligned} \tag{95}$$

and

$$\begin{aligned}\mathscr{P}_{13}(\delta u, u) = {} & \mathscr{P}_{13}(\delta u, u_c) + \xi[3a\lambda_c \mathscr{P}_{112}(\delta u, u_c', u_c) + 3\mathscr{P}_{112}(\delta u, u_1, u_c)] \\ & + \xi^2[3(a\lambda_c)^2 \mathscr{P}_{112}(\delta u, u_c, u_c') + 6a\lambda_c \mathscr{P}_{1111}(\delta u, u_c, u_c', u_1) \\ & + 3\mathscr{P}_{112}(\delta u, u_c, u_1) + 3b\lambda_c \mathscr{P}_{112}(\delta u, u_c', u_c) \\ & + \tfrac{3}{2}(a\lambda_c)^2 \mathscr{P}_{112}(\delta u, u_c'', u_c) + 3\mathscr{P}_{112}(\delta u, u_2, u_c)] \\ & + \xi^3[6ab\lambda_c^2 \mathscr{P}_{112}(\delta u, u_c, u_c') + 3(a\lambda_c)^3 \mathscr{P}_{1111}(\delta u, u_c, u_c', u_c'') \\ & + 6a\lambda_c \mathscr{P}_{1111}(\delta u, u_c, u_c', u_2) + 6b\lambda_c \mathscr{P}_{1111}(\delta u, u_c, u_c', u_1) \\ & + 3(a\lambda_c)^2 \mathscr{P}_{1111}(\delta u, u_c, u_c'', u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u_1, u_2) \\ & + 3c\lambda_c \mathscr{P}_{112}(\delta u, u_c', u_c) + 3ab\lambda_c^2 \mathscr{P}_{112}(\delta u, u_c'', u_c) \\ & + \tfrac{1}{2}(a\lambda_c)^3 \mathscr{P}_{112}(\delta u, u_c''', u_c) + 3\mathscr{P}_{112}(\delta u, u_3, u_c) + (a\lambda_c)^3 \mathscr{P}_{13}(\delta u, u_c') + \mathscr{P}_{13}(\delta u, u_1) \\ & + 3(a\lambda_c)^2 \mathscr{P}_{112}(\delta u, u_1, u_c') + 3a\lambda_c \mathscr{P}_{112}(\delta u, u_c', u_1)] + O(\xi^4)\end{aligned} \tag{96}$$

Analogous to the procedure in Appendix B comparison of terms of like order in ξ in the Taylor expansions provide the following formulas:

$$\begin{aligned}\delta p_1 = {} & -a\lambda_c \delta p_c' + a\lambda_c[\mathscr{P}_{11}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c', u_c)] \\ & + \mathscr{P}_{11}(\delta u, u_1) + 2\mathscr{P}_{111}(\delta u, u_c, u_1) + 3\mathscr{P}_{112}(\delta u, u_1, u_c)\end{aligned} \tag{97}$$

$$\begin{aligned}\delta p_2 = {} & -b\lambda_c \delta p_c' - \tfrac{1}{2}a^2\lambda_c^2 \delta p_c'' + a\lambda_c[2\mathscr{P}_{111}(\delta u, u_c', u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u_c', u_1)] \\ & + b\lambda_c[\mathscr{P}_{11}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c', u_c)] \\ & + \tfrac{1}{2}(a\lambda_c)^2[\mathscr{P}_{11}(\delta u, u_c'') + 2\mathscr{P}_{12}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c'') \\ & + 6\mathscr{P}_{112}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c'', u_c)] \\ & + \mathscr{P}_{11}(\delta u, u_2) + \mathscr{P}_{12}(\delta u, u_1) + 2\mathscr{P}_{111}(\delta u, u_c, u_2) \\ & + 3\mathscr{P}_{112}(\delta u, u_c, u_1) + 3\mathscr{P}_{112}(\delta u, u_2, u_c)\end{aligned} \tag{98}$$

$$\begin{aligned}
\delta p_3 = & -c\lambda_c \delta p_c' - ab\lambda_c^2 \delta p_c'' - \tfrac{1}{6}a^3\lambda_c^3 \delta p_c''' \\
& + a\lambda_c[2\mathscr{P}_{111}(\delta u, u_c', u_2) + 6\mathscr{P}_{1111}(\delta u, u_c, u_c', u_2) + 3\mathscr{P}_{112}(\delta u, u_c', u_1)] \\
& + b\lambda_c[2\mathscr{P}_{111}(\delta u, u_c', u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u_c', u_1)] \\
& + c\lambda_c[\mathscr{P}_{11}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c', u_c)] \\
& + \tfrac{1}{2}(a\lambda_c)^2[2\mathscr{P}_{111}(\delta u, u_c'', u_1) + 6\mathscr{P}_{1111}(\delta u, u_c, u_c'', u_1) + 6\mathscr{P}_{112}(\delta u, u_1, u_c')] \\
& + ab\lambda_c^2[\mathscr{P}_{11}(\delta u, u_c'') + 2\mathscr{P}_{12}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c'') \\
& + 6\mathscr{P}_{112}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c'', u_c)] \\
& + \tfrac{1}{6}(a\lambda_c)^3[\mathscr{P}_{11}(\delta u, u_c''') + 6\mathscr{P}_{111}(\delta u, u_c', u_c'') + 2\mathscr{P}_{111}(\delta u, u_c, u_c''') \\
& + 18\mathscr{P}_{1111}(\delta u, u_c, u_c', u_c'') + 3\mathscr{P}_{112}(\delta u, u_c''', u_c) + 6\mathscr{P}_{13}(\delta u, u_c')] \\
& + \mathscr{P}_{11}(\delta u, u_3) + 2\mathscr{P}_{111}(\delta u, u_1, u_2) + 2\mathscr{P}_{111}(\delta u, u_c, u_3) \\
& + 6\mathscr{P}_{1111}(\delta u, u_c, u_1, u_2) + 3\mathscr{P}_{112}(\delta u, u_3, u_c) + \mathscr{P}_{13}(\delta u, u_1)
\end{aligned} \tag{99}$$

After introduction of:

$$\begin{aligned}
\delta p_c &= \mathscr{P}_1(\delta u) + \mathscr{P}_{11}(\delta u, u_c) + \mathscr{P}_{12}(\delta u, u_c) + \mathscr{P}_{13}(\delta u, u_c) \\
\delta p_c' &= \mathscr{P}_{11}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c', u_c) \\
\delta p_c'' &= \mathscr{P}_{11}(\delta u, u_c'') + 2\mathscr{P}_{12}(\delta u, u_c') + 2\mathscr{P}_{111}(\delta u, u_c, u_c'') \\
&\quad + 6\mathscr{P}_{112}(\delta u, u_c, u_c') + 3\mathscr{P}_{112}(\delta u, u_c'', u_c) \\
\delta p_c''' &= \mathscr{P}_{11}(\delta u, u_c''') + 6\mathscr{P}_{111}(\delta u, u_c', u_c'') + 2\mathscr{P}_{111}(\delta u, u_c, u_c''') \\
&\quad + 6\mathscr{P}_{13}(\delta u, u_c') + 18\mathscr{P}_{1111}(\delta u, u_c, u_c', u_c'') + 3\mathscr{P}_{112}(\delta u, u_c, u_c''')
\end{aligned} \tag{100}$$

the expressions for δp_1, δp_2 and δp_3 are reduced to (22).

APPENDIX D

Modified principle of virtual displacements and its derivatives at bifurcation

The modified principle of virtual displacements and its derivatives up to order three at the point of bifurcation, i.e. at $\lambda = \lambda_c$, are used to eliminate terms in the buckling and postbuckling problems.

D.1. Modified principle of virtual displacements.

$$0 = \sigma_c \cdot \delta\varepsilon_c - \lambda_c \delta B_c - C_c \cdot \delta\eta - \eta_c \cdot \delta C_c \tag{101}$$

D.2. First derivative.

$$0 = \sigma_c' \cdot \delta\varepsilon_c + \sigma_c \cdot \delta\varepsilon_c' - \delta B_c - \lambda_c \delta B_c' - C_c' \cdot \delta\eta - \eta_c' \cdot \delta C_c - \eta_c \cdot \delta C_c' \tag{102}$$

D.3. Second derivative.

$$0 = 2\sigma_c' \cdot \delta\varepsilon_c' + \sigma_c \cdot \delta\varepsilon_c'' + \sigma_c'' \cdot \delta\varepsilon_c - 2\delta B_c' - \lambda_c \delta B_c'' - C_c'' \cdot \delta\eta - 2\eta_c' \cdot \delta C_c' - \eta_c'' \cdot \delta C_c - \eta_c \cdot \delta C_c'' \tag{103}$$

D.4. Third derivative.

$$\begin{aligned}
0 = 3\sigma_c'' \cdot \delta\varepsilon_c' + 3\sigma_c' \cdot \delta\varepsilon_c'' + \sigma_c \cdot \delta\varepsilon_c''' + \sigma_c''' \cdot \delta\varepsilon_c - 3\delta B_c'' - \lambda_c \delta B_c''' \\
- C_c''' \cdot \delta\eta - 3\eta_c' \cdot \delta C_c'' - 3\eta_c'' \cdot \delta C_c' - \eta_c''' \cdot \delta C_c - \eta_c \cdot \delta C_c'''
\end{aligned} \tag{104}$$